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On the evolution of thermal disturbances during natural convection in a porous medium

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In natural convection in a porous medium heated from below, the convective flow in two dimensions becomes unsteady above a certain critical Rayleigh number and exhibits a fluctuating or oscillatory behaviour (depending on the confinement in the horizontal dimension). This fluctuating behaviour is due to a combination of the instability of the thermal boundary layers at horizontal boundaries together with a 'triggering' effect of earlier disturbances. The point of the origin of the instability of the thermal boundary layer appears to play a dominant role in determining the regularity of the fluctuating flow. This numerical study investigates the importance of this point of evolution and concludes that there may exist more than one oscillatory mode of convection, depending on its position. The investigation focuses on the symmetry of the flow and demonstrates that with stable and accurate numerical schemes, an artificial symmetry may be imposed in the absence of realistic physical noise. If an initially symmetric perturbation is imposed the flow retains an essentially symmetric flow pattern with a high degree of regularity in the oscillatory behaviour. The imposition of an asymmetric perturbation results in a degradation of regularity. The appearance of the symmetric, regularly oscillatory flow is characterized by a symmetric (and stationary) arrangement of the points of origin of the instability of the upper and lower thermal boundary layers; in the case of the irregular oscillations the points of origin are not symmetric and their locations are not fixed.

1. Introduction

In consideration of natural convection through a confined porous medium heated from below, it was observed originally by Combarnous & LeFur (1969) that above a critical Rayleigh number R_2 a second mode of convection occurred. This convective regime produced a different Nusselt-number or Rayleigh-number dependence to the original convective disturbance that occurs above the first critical Rayleigh number R_1 ($4\pi^2$ for an infinitely wide layer, see Lapwood 1948). The higher mode was then observed experimentally by Caltagirone, Cloupeau & Combarnous (1971) to be a fluctuating convective state, which showed a permanently unsteady behaviour as thermal anomalies formed, dissipated, and reformed. The same behaviour was observed in different experiments by Horne & O'Sullivan (1974) who also noted com-

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parable effects in numerical solutions of the equations governing the flow. Further numerical representations were obtained by Caltagirone (1974, 1975), who also went on to determine the value of R_2 using the method of weighted residuals (Finlayson 1972). The properties of R_2 have also been examined by Straus (1974).

The mechanism of these effects lies in the instability of the thermal boundary layers over the heated or cooled horizontal boundaries. Horne & O'Sullivan (1978) examined the behaviour of a convecting region with only one boundary layer in order to isolate one set of disturbances from the other. By forcing a loss of identity to each thermal anomaly in this way they concluded that the unstable boundary layer evolved 'thermals' in accordance with the concept of Howard (1964) and as observed in the experiments of Sparrow, Husar & Goldstein (1970). It was determined however, that the cyclic 'triggering' mechanism also influenced the behaviour of the flow by instilling regularity. The concept of cyclic interaction was demonstrated originally by Welander (1967) and Keller (1966) for periodic convective flows in fluid loops. These fluid loop instabilities were later seen in the experiments of Creveling *et al.* (1975). Moore & Weiss (1973) suggested that the disturbances that they observed in the case of a convecting fluid layer (without the porous medium) were due to triggering of disturbances by their predecessors that had circulated around the cell. Krishnamurti (1970) also observed oscillation in the convecting-fluid layer problem and examined the significance of both thermal boundary-layer instability and cyclic triggering. Since it was determined that the Rayleigh and Prandtl numbers were well below those necessary for thermal instability of the boundary layer, it was concluded, as in Moore & Weiss (1975), that the effect was due to cyclic triggering. Horne & O'Sullivan (1978) report that the thermal boundary layer in the porous medium case *is* at a Rayleigh number sufficient to become unstable and hence conclude that the oscillatory behaviour in this case is due to a combination of both triggering and thermal boundary-layer instability. Examination of the Rayleigh-number/fluctuation-period (R/τ_p) dependence allows exclusion of the thermal boundary layer as the only mechanism of the behaviour. The observed dependence is

$$\tau_p \sim R^{-\frac{3}{2}},$$

a somewhat longer time than R^{-2} which would be expected if the boundary layer had only to grow to critical thickness before a disturbance was formed. The $\frac{3}{2}$ power relationship is more characteristic of the circulation time (which varies to the power $-\frac{1}{2}$) divided by the number of disturbances circulating (which varies as R), and hence is indicative of triggering. The effects of triggering are of course governed by the influence of the confining 'box' boundary conditions, and this argument may not apply to the unphysical 'infinite-layer' case.

In this investigation we examine the regularity of the fluctuating behaviour again in a different way, since the formation of these oscillations remains of fundamental interest in the understanding of convective instabilities in boxes. The study of Horne & O'Sullivan (1974) showed that both regular and irregular flows are possible and suggested that regularity could be induced by restricting the location of the point of origin of the disturbances. In the light of more recent work comparing thermal boundary-layer instability and cyclic triggering, we examine the significance of the point of origin more closely. Introducing temperature anomalies at different points along one of the boundaries, the most likely point of evolution can be determined as a point which is downstream of any perturbation causing an interruption to the regularity of the flow. A perturbation on or downstream of the evolution point should not affect the gestation of disturbances.

The location of the point of first instability of the thermal boundary layer has thus been isolated (at least in cases in which it does not move unduly). In some cases this location is not permanent, and moves with each new disturbance, causing a degradation in regularity. However, in such cases an oscillatory flow may still exist in the form of a repeating sequence of temperature changes. Introduction of a fixed perturbation in this case causes another degradation in regularity, which suggests that the repeating sequence arises from disturbances forming at more than one location.

2. Governing equations and boundary conditions

Assuming the validity of Darcy's law and the Boussinesq approximation and neglecting inertial effects, the governing equations for convective flow through porous medium are

$$\nabla^2\psi = -\partial\theta/\partial x \tag{1}$$

and

$$\frac{\partial\theta}{\partial t} = \nabla^2\theta - R\left[\frac{\partial\psi}{\partial y}\frac{\partial\theta}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\theta}{\partial y}\right], \tag{2}$$

where x, y, and t are the non-dimensional space and time co-ordinates (y vertically upward), and θ and ψ are the non-dimensional temperature and stream function respectively. The Rayleigh number R is defined as

$$R = \frac{gak\Delta T\alpha}{\kappa\nu}\lambda, \tag{3}$$

where a is the depth of the porous layer, k is the permeability, κ is the thermal diffusivity of the fluid-saturated medium, ν is the kinematic viscosity, and α the coefficient of thermal expansion of the fluid, ΔT is the temperature differential across the region, λ is the ratio of the volumetric heat capacity of the fluid to that of the saturated formation and g is the acceleration due to gravity.

The characteristic length and time used in non-dimensionalizing are a and a^2/κ respectively. The non-dimensional temperature is defined as:

$$\theta = \frac{T - T_0}{T_1 - T_0}, \tag{4}$$

where T_0 and T_1 are the maximum and minimum boundary temperatures.

The choice of boundary configuration is extremely important since the presence of confining boundaries strongly affects the form of the oscillatory solution. By restricting the wavelength of any flow pattern to lie within two confining vertical boundaries, the 'box' configuration permits the persistence of a flow at a particular wavelength that would have been unstable in favour of another wavelength, had it been possible for the more favoured to 'fit' between the boundaries. Thus, to isolate the influence of the boundaries, a single aspect ratio is selected and only a square cell is considered. The boundary conditions are kept as simple as possible, i.e. impermeable boundaries on all sides, constant (high) temperature boundary at the bottom, constant (low) temperature boundary at the top, and adiabatic boundaries to the sides. This con-

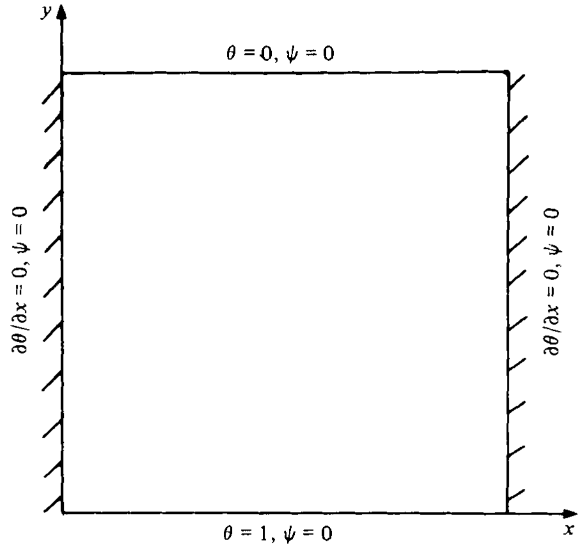

FIGURE 1. Boundary conditions.

figuration is the one most frequently investigated experimentally and numerically so far, and allows the problem to be examined with reference to the greatest volume of former work. The problem boundaries are illustrated in figure 1.

In order to investigate the position of evolution of the instability of the thermal boundary layer, a perturbation is introduced at the constant temperature lower boundary. This is achieved simply, by raising the temperature at a single point by 10 % (i.e. to $\theta = 1{\cdot}1$). In analytical terms this would correspond to a 'spike' at that point. However since the solution is achieved using a numerical solution on a finite difference mesh the perturbation should be viewed rather as a 'triangle'.

The initial conditions are obtained using a 'stirred' flow. Given an initially cellular motion in the flow region of

$$\psi = f_1 \sin m\pi x \sin n\pi y, \tag{5}$$

then for zero net convective transfer the temperature distribution from (1) will be

$$\theta = (1-y) - \frac{f_1 \pi(m^2+n^2)}{m} \cos m\pi x \sin n\pi y. \tag{6}$$

To obtain an initial motion that is asymmetric with respect to the hot and cold boundaries, the initial conditions can be specified as the superposition of two forcing functions as in equations (5) and (6), with different values of m in each (at least one of which should be even).

3. Numerical solution

Two separate numerical procedures are used here with essentially comparable results. The first uses the methods described in Horne & O'Sullivan (1974, 1978), namely the non-iterative, odd–even reduction technique (Busbee, Golub & Nielson 1970) for the solution of the stream-functioning equation (1), and fourth-order Arakawa differencing (Arakawa 1966) for the representation of the advection terms in the energy equation (2). The space differencing is fourth-order accurate right to the boundaries, as in Horne & O'Sullivan (1978). The accuracy of the time differencing has been improved by using an approximate form of the Crank–Nicolson technique obtained by combining a forward time differencing step and a leap-frog differencing step. The temperature and stream-function distribution at a half time-step forward are calculated using forward differencing, then these values used to recalculate the rate of change at the half time step point. This rate of change is then used to determine the temperature and stream function one full time step ahead of the original one. This technique avoids the 'time-splitting' instability encountered with leap-frog differencing by itself (Orszag & Israeli 1974).

The second procedure involves methods previously used in Caltagirone (1974, 1975), namely the alternating direction implicit (ADI) method for the solution of both the energy and stream-function equations, forming each in terms of an unknown Laplacian using central differences. To maintain accuracy of the time difference, this method solves the energy equation iteratively.

The mesh size for the first procedure has to be one more than a power of two (e.g. 33×33 or 65×65), for Rayleigh numbers around 500, a 33×33 mesh produces identical results, globally and locally to a 65×65 mesh. The second procedure produces comparable results with a 48×48 mesh. Time steps used are of order 0·0002 with the non-dimensional time as defined in § 2.

The details of these two methods have been described previously and are not repeated here. Both methods have demonstrated success in reproducing experimentally observed flows (Horne & O'Sullivan 1974; Caltagirone 1974). One motivation for conducting this joint study was to compare results obtained with substantially different numerical methods. The observed agreement builds confidence in the numerical solutions, and the cooperative study resulted in worthwhile streamlining of both methods of attack. It should be emphasized that the appearance of irregular fluctuating flows in an already nonlinear problem precludes all current analytical techniques and leaves physical and numerical experiments as the only alternatives.

4. Results

The initial purpose of this study was to pinpoint the location of the evolution of the instability in the thermal boundary layer, i.e. the point of origin of a 'thermal' in the sense used by Sparrow, Husar & Goldstein (1970). It is practically impossible to observe the actual birth of an incipient thermal since it must reach a certain magnitude before becoming apparent, therefore it is necessary to resort to the indirect method, perturbing the boundary layer in the vicinity of the anticipated location of the gestation and observing the effect.

An early impression of previously determined flows suggests that in an essentially unicellular oscillatory flow, the thermals originate symmetrically from the centre of the upper and lower boundaries. To investigate this matter it is necessary to choose a Rayleigh number at which the solution is unicellularly oscillatory (in the square cell): it must therefore be above 390 in order to be oscillatory but below 1000 to avoid the upset of the solution into multicellular modes. A value of 500 was selected for the set of numerical experiments. A temperature anomaly was positioned at points $x = \frac{1}{4}, \frac{3}{8}, \frac{1}{2}, \frac{5}{8}$ and $\frac{3}{4}$ along the lower (heated) boundary and the flow patterns compared to the

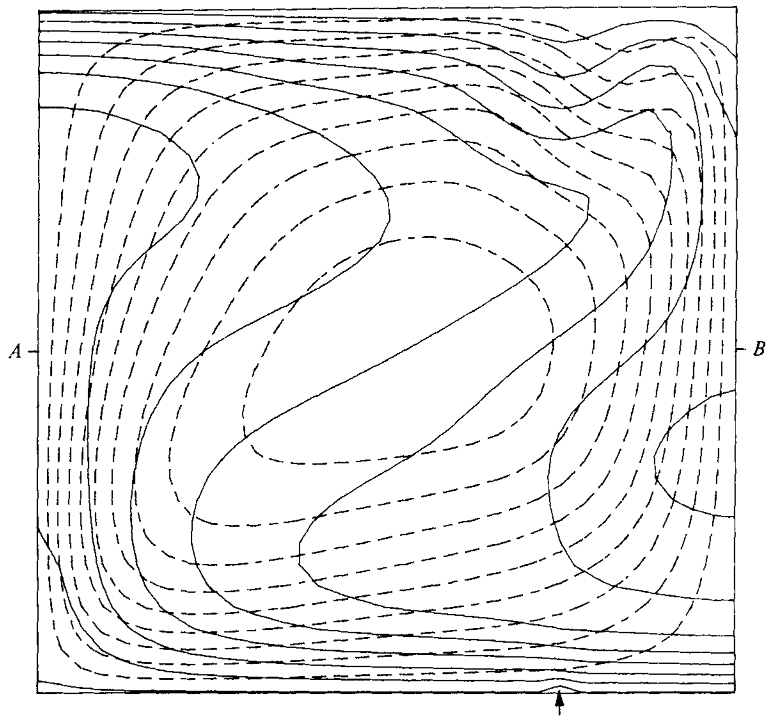

FIGURE 2. Asymmetric flow pattern caused by the introduction of a boundary temperature perturbation at the point ($\frac{3}{4}$, 0) into an initially symmetric flow. Rayleigh number is 500, solid lines are isotherms, broken lines are streamlines.

Position of anomaly	Arrival time at (A) Mean	Arrival time at (A) Standard deviation	Arrival time at (B) Mean	Arrival time at (B) Standard deviation
no anomaly	0·00973	0·00071	0·00973	0·00071
$\frac{1}{4}$	0·00979	0·00060	0·00975	0·00056
$\frac{3}{8}$	0·00969	0·00079	0·00969	0·00084
$\frac{1}{2}$	0·00970	0·00062	0·00970	0·00062
$\frac{5}{8}$	0·00964	0·00086	0·00994	0·00130
$\frac{3}{4}$	0·00945	0·00236	0·00965	0·00210

TABLE 1. Mean and standard deviations of arrival times at positions (A) in ascending flow, (B) in descending flow, at a Rayleigh number of 500

case where there was no temperature anomaly. The initial motion imposed on the flow was symmetric.

In every case the flow exhibited the unicellular fluctuating behaviour, periodically emitting 'thermals' from the upper and lower boundary layers. The nature of this flow is illustrated in figure 2. However the regularity of the flow differed depending on the position of temperature anomaly on the boundary. This variation in regularity can best be seen by comparing the statistics of the series of the time periods between successive thermals. Choosing a fixed point in the path of the 'thermals', the passage of a 'thermal' is marked as a relative maximum in temperature in the rising flow or a relative minimum in temperature in the descending flow. Table 1 summarizes the

	(i)	(ii)	(iii)	(iv)
	0·0099	0·0096	0·0099	0·0108
	0·0096	0·0099	0·0102	0·0090
	0·0093	0·0093	0·0096	0·0087
	0·0108	0·0108	0·0099	0·0117
	0·0090	0·0090	0·0093	0·0084
	0·0096	0·0096	0·0099	0·0096
Mean	0·0097	0·0097	0·0098	0·0097
Standard deviation	0·00062	0·00062	0·00031	0·00130

TABLE 2. Time intervals between successive extreme in (i) temperature at (A), (ii) temperature at (B), (iii) Nusselt number, (iv) maximum stream function. Rayleigh number 500, anomaly at $x = \frac{1}{2}$

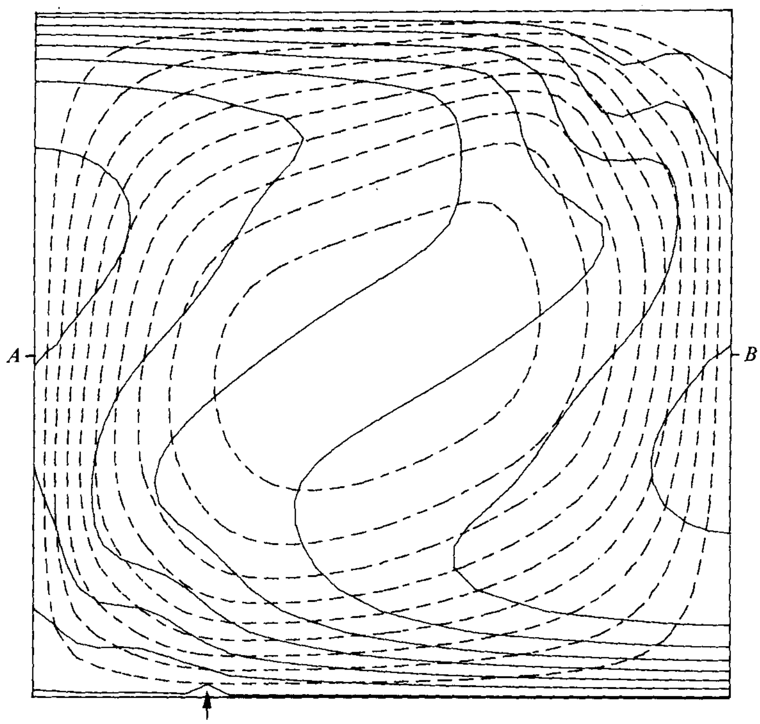

FIGURE 3. As in figure 2 but with boundary temperature perturbation at the point ($\frac{1}{4}$, 0). In this case symmetry persists.

mean and standard deviation of the time separation between arrivals at each of two points in the flow, one in the ascending flow and one in the descending flow. These two points are indicated in figure 2. There are also other means of signalling the 'flight' of a 'thermal', since each is accompanied by an increase in the maximum stream function in the region, and the arrival of a 'thermal' at the upper boundary causes a relative maximum in the Nusselt number at that boundary. Table 2 shows typical sequences of time intervals between successive 'thermals', measured in four separate ways. In this particular case the anomaly is at $x = \frac{1}{2}$, and it is seen that although not

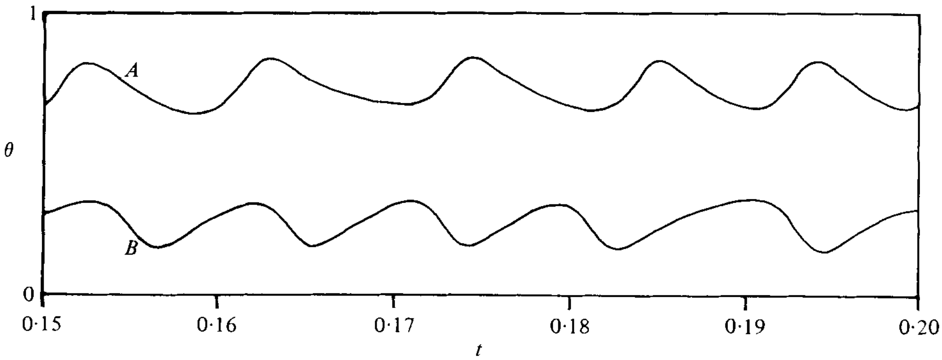

FIGURE 4. Oscillation of temperatures at points A and B. Asymmetrically initiated flow, Rayleigh number 500, no boundary temperature perturbations.

completely regular, there is only a very small deviation from the mean arrival interval (variations of less than 0·000 25 are not significant since this is the resolution of the time step).

It is observed in this first set of results that the time sequence of 'thermal' evolutions is essentially unaffected by temperature anomalies at the boundary which lie at $x = \frac{1}{2}$ or downstream of that position. On the other hand, when the anomaly is upstream of $x = \frac{1}{2}$, the symmetry between the ascending and descending 'thermals' is lost, the mean time interval is altered, and the deviation of the intervals from the mean increases. Figure 2 shows the loss of symmetry caused by the anomaly at $x = \frac{3}{4}$ (i.e. an upstream anomaly). This may be compared with figure 3 in which the anomaly is at $x = \frac{1}{4}$ (i.e. downstream); in this case the symmetry persists.

The suggestion is therefore that 'thermals' originate at $x = \frac{1}{2}$ on the upper and lower boundaries.

In this first set of results the flows are originally set in motion by an induced movement which is symmetric and, except in the cases noted already, remain symmetric throughout a long period of time (up to 20 oscillations were calculated). It is a property of numerical solutions that noise is less than in physical or experimental situations, and it is therefore not known whether this persistence of symmetry is physically realistic or is an artifice of the noiseless numerical calculation together with the 'perfect' symmetric initial condition. This is particularly significant where explicit and direct solution techniques are used since even the errors are propagated symmetrically. This question may be investigated by repeating the experiments with an asymmetrically started motion.

Starting with an initial perturbation that is not symmetric in x, the flow remains asymmetric, with widely varying fluctuating periods in both the rising and descending portions of the flow. For example, for an asymmetrically started flow with no temperature anomaly, the time period of successive temperature maxima has a mean value of 0·009 67 and a standard deviation of 0·001 78 at the point in the ascending flow, and mean and standard deviation of 0·009 70 and 0·001 63 respectively in the descending flow. However, there is a strong *negative* correlation between the rising and falling disturbances, for example, the rising 'thermal' shows cycles of two shorter arrival times and three longer ones, while the falling 'thermal' shows cycles of two longer arrival times and three shorter ones. This behaviour is illustrated in figure 4.

In other cases, using a different initial asymmetric disturbance, the flow takes up the usual, more or less regular cycle.

In these asymmetrically started cases, introducing a temperature anomaly at the mid-point of the heated lower boundary affects the flow considerably. The disturbance continues for a long period of time (more than 0·2) without forming into a regular pattern, and has time periods in the ascending flow with mean and standard deviation 0·0110 and 0·0036 respectively; in the descending flow these values are 0·0122 and 0·0032.

Extending the analysis to other Rayleigh numbers, at the lower Rayleigh number of 400 a symmetrically initiated flow shows only a small amplitude oscillation (a change in temperature of about 0·02 at point A, and a variation in Nusselt number between 5·32 and 5·23) with a mean period of 0·0115. On the other hand, an asymmetrically started flow has a much larger amplitude oscillation (temperature change of order 0·06, Nusselt number varying between 5·50 and 5·23) with a mean period of 0·0140. At the higher Rayleigh number of 700, a symmetrically initiated flow quickly loses its symmetry and settles into an oscillatory flow with mean period 0·054 and Nusselt number variation between 9·46 and 6·94. On the other hand an asymmetrically initiated flow separates into two cells of differing size, the larger one oscillating with mean period 0·0060 and the smaller one steady. The Nusselt number varies between 7·8 and 8, but the amplitude decreases to settle on a value in the vicinity of 7·7 as the cells equilibrate in size and the oscillations decrease. It is worthy of note that there is no discernable change in the period of the oscillations as the cell decreases in size and the oscillations reduce in amplitude.

5. Conclusions

In a flow which is initially symmetric, undergoing unicellular oscillatory convection in a closed box, the location of the initiation of instability of the thermal boundary layers, giving rise to the evolution of 'thermals', takes place half way along the boundary. The symmetry of the flow can be upset by perturbations upstream of this point. In the case of an asymmetric flow the point of initiation is not fixed and the regularity of the oscillatory convection is degraded. However there may exist another regular sequence based on a series of 'thermals' that repeat after a certain number (five for a Rayleigh number of 500). The introduction of a fixed perturbation in such a case interrupts the sequence of initiation locations and prevents the oscillations from ever becoming regular.

It has also been seen that there exist more than one stable set of cyclic disturbances – two each have been observed for the Rayleigh numbers 400, 500 and 700. With the exception of the Rayleigh number 700 in which no persistently symmetrical flow was observed, the more symmetric of the two alternatives had a shorter time period and a smaller amplitude in both Nusselt number and temperature variation at a point. It is clear then that the symmetric arrangement of points of gestation of 'thermals' allows the shortest path for one 'thermal' to trigger its successor, resulting in an early flight and consequent smaller magnitude of disturbance. The appearance of either one flow or the other depends on the original movement in the region, another example of the dependence of nonlinear flows on initial conditions, as noted previously by Horne & O'Sullivan (1974) and also by Straus & Schubert (1979). The difference in mechanism between the alternative cycles arises from differences in the location of the initiation points of 'thermals', and the number and complexity of alternatives increases with Rayleigh number as the supercritical extent of the thermal boundary layer increases. The different cycles appear stable to internal perturbations but can be interrupted externally (in this case by introducing the temperature anomaly at the boundary). These alternative flows might therefore be classed as 'weak' alternatives as opposed to the 'strong' alternatives, steady or oscillatory (Horne & O'Sullivan 1974) and two dimensional or three dimensional (Horne 1979) which cannot be so easily perturbed. The appearance of symmetrical and non-symmetrical alternatives has also been noted in a recent paper by Straus & Schubert (1980).

It is worthwhile noting the difficulty that arises in the mathematical solution of this problem. This nonlinear problem has a plethora of possible alternative flow regimes and histories depending on the conditions applied initially and subsequently. Therefore the 'too perfect' conditions that are achieved using analytical or numerical techniques (paradoxically the most accurate ones in particular) may give rise to other artificial solutions that are divorced from the flows observed in 'noisy' physical systems. Another example of this is easily observed in the simpler onset of a convection problem – the numerical techniques used here will hold a conduction solution for indefinite periods at Rayleigh numbers well above critical in the absence of the small perturbation necessary for the onset of a convective flow. It is perhaps time to admit that mathematical solutions to nonlinear problems must of necessity include non-deterministic forcing effects in order to avoid solutions mathematically correct but physically unlikely.

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REFERENCES

Arakawa, A. 1966 *J. Comp. Phys.* **1**, 119.

Busbee, B. L., Golub, C. H. & Nielson, D. W. 1970 *SIAM J. Numerical Anal.* **7**, 627.

Caltagirone, J. P. 1974 *C. R. Acad. Sci. Paris* **278**, 259.

Caltagirone, J. P. 1975 *J. Fluid Mech.* **72**, 269.

Caltagirone, J. P., Cloupeau, M. & Combarnous, M. A. 1971 *C. Acad. Sci. Paris* B **273**, 883.

Combarnous, M. F. & LeFur, B. 1969 *C. R. Acad. Sci. Paris* B **269**, 1009.

Creveling, H. F., DePaz, J. F., Baladi, J. Y. & Schoenhals, R. J. 1975 *J. Fluid Mech.* **67**, 65.

Finlayson, B. A. 1972 *The Method of Weighted Residuals and Variational Principles*. Academic.

Horne, R. N. 1979 *J. Fluid Mech.* **92**, 751.

Horne, R. N. & O'Sullivan, M. J. 1974 *J. Fluid Mech.* **66**, 339.

Horne, R. N. & O'Sullivan, M. J. 1978 *Phys. Fluids* **21**, 1260.

Howard, L. N. 1964 *Proc. 11th Int. Cong. on Appl. Mech., Munich*, p. 1109.

Keller, J. B. 1966 *J. Fluid Mech.* **26**, 599.

Krishnamurti, R. 1970 *J. Fluid Mech.* **42**, 309.

Lapwood, E. R. 1948 *Proc. Camb. Phil. Soc.* **44**, 508.

Moore, D. R. & Weiss, N. P. 1973 *J. Fluid Mech.* **72**, 269.

Orszag, S. A. & Israeli, M. 1974 *Ann. Rev. Fluid Mech.* **6**, 281.

SPARROW, E. M., HUSAR, R. B. & GOLDSTEIN, R. J. 1970 *J. Fluid Mech.* **41**, 793.

STRAUS, J. M. 1974 *J. Fluid Mech.* **64**, 51.

STRAUS, J. M. & SCHUBERT, G. E. 1979 *J. Fluid Mech.* **91**, 155.

STRAUS, J. M. & SCHUBERT, G. E. 1980 Modes of finite-amplitude, three-dimensional convection in rectangular boxes of fluid-saturated porous material. *J. Fluid Mech.* (to appear).

WELANDER, P. 1967 *J. Fluid Mech.* **29**, 17.